VISCOUS FLUID FLOW IN A HORIZONTAL LAYER DUE TO THERMAL INHOMOGENEITY OF THE LOWER BOUNDARY

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The nature of the flow induced in a layer by thermal inhomogeneity of one of the boundaries is determined on the basis of a numerical solution of the complete Navier–Stokes equations. It is found that stable stratification effectively suppresses the generated flow.

We wish to consider viscous incompressible fluid flow in the Boussinesq approximation. The motion takes place in a flat horizontal layer, the lower boundary of which comprises a solid surface. The temperature of this surface is T_1 everywhere except for one section that has a temperature T_2. The upper boundary of the layer is a free surface, which is fixed in space and on which a temperature T_3 is specified everywhere.

In the Boussinesq approximation (conventional notation) the flow is described by the system of equations

$$\begin{gathered}\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}=\nu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial x},\\ \frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}=\nu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial y}+\beta g(T_1-T),\\ \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0,\\ \frac{\partial T}{\partial t}+v_x\frac{\partial T}{\partial x}+v_y\frac{\partial T}{\partial y}=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)\end{gathered}\tag{1}$$

subject to the boundary conditions

$$\begin{gathered}v_x=0,\quad v_y=0,\quad T=T_1\quad\text{for}\quad y=0,\ |x|>a,\\ v_x=0,\quad v_y=0,\quad T=T_2\quad\text{for}\quad y=0,\ |x|\leqslant a,\\ \frac{\partial v_x}{\partial y}=0,\quad v_y=0,\quad T=T_3\quad\text{for}\quad y=h.\end{gathered}$$

A periodicity condition is given on the side boundaries of the layer. We introduce dimensionless variables according to the equations

$$u=\frac{v_x h}{\nu},\quad v=\frac{v_y h}{\nu},\quad \vartheta=\frac{T-T_1}{T_2-T_1},\quad P=\frac{ph^2}{\rho\nu^2},\quad \bar{x}=\frac{x}{h},$$
$$\bar{y}=\frac{y}{h},\quad \bar{t}=\frac{t\nu}{h^2}.$$

The system of equations takes the form (we drop the overbar from dimensionless quantities from now on)

$$\begin{gathered}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial P}{\partial x},\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}-\frac{\partial P}{\partial y}-\mathrm{Gr}\,\vartheta,\end{gathered}$$

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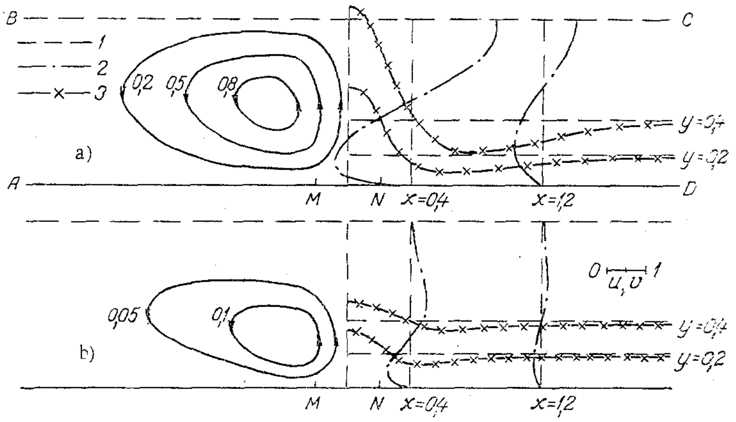

Fig. 1. Streamlines and velocity profiles. a) $\gamma = 0$; b) $\gamma = 10$; 1) streamlines; 2) velocity u; 3) velocity v.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$
$$\frac{\partial \vartheta}{\partial t} + u\frac{\partial \vartheta}{\partial x} + v\frac{\partial \vartheta}{\partial y} = \frac{1}{\mathrm{Pr}}\left(\frac{\partial^2 \vartheta}{\partial x^2} + \frac{\partial^2 \vartheta}{\partial y^2}\right) \tag{2}$$

with the boundary conditions

$$u = 0, \quad v = 0, \quad \vartheta = 0 \quad \text{for} \quad y = 0, \ |x| > a/h,$$
$$u = 0, \quad v = 0, \quad \vartheta = 1 \quad \text{for} \quad y = 0, \ |x| \leqslant a/h,$$
$$\frac{\partial u}{\partial y} = 0, \quad v = 0, \quad \vartheta = \gamma \quad \text{for} \quad y = 1.$$

Here $\mathrm{Gr} = g\beta(T_2 - T_1)h^3/\nu^2$ is the Grashof number, $\mathrm{Pr} = \nu/k$ is the Prandtl number, and $\gamma = (T_3 - T_1)/(T_2 - T_1)$.

In this article we give results obtained by the relaxation method for the case of steady flow.

Introducing the stream function ψ and the vorticity φ according to the equations $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$, $\varphi = \partial u/\partial y - \partial v/\partial x$, we arrive at the system

$$\frac{\partial \varphi}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \varphi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \varphi}{\partial y} = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} - \mathrm{Gr}\frac{\partial \vartheta}{\partial x},$$
$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \varphi, \tag{3}$$
$$\frac{\partial \vartheta}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \vartheta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \vartheta}{\partial y} = \frac{\partial^2 \vartheta}{\partial x^2} + \frac{\partial^2 \vartheta}{\partial y^2}.$$

We take the Prandtl number equal to unity. The boundary conditions for the stream function and the vorticity have the form $\psi = 0$ at $y = 0$ and $\psi = 0$, $\varphi = 0$ at $y = 1$. An additional condition for the vorticity at the solid boundary is formulated in the finite-difference solution of the equations [1].

We seek a solution of the system (3) by the finite-difference method according to an "explicit-implicit" scheme; the procedure is similar to that used in [2, 3]. Accordingly, the finite-difference solution of the first and third equations of the system (3) is determined by an explicit scheme, and in each step the difference analog of the second equation of the system (3) is solved by the Seidel iterative procedure. The size of the time step Δt is constrained by the stability condition for the explicit scheme; if the stability condition is violated, then the number of iterations required in order to find the values of the stream function at the nodes of the computing grid (within specified error limits) increases. This fact provides a basis for automatic selection of the step Δt, which is increased as the number of iterations is decreased and vice versa.

Below, we give the results obtained for various relations between the temperatures T_1, T_2, and T_3.

If $T_3 = T_1$ ($\gamma = 0$), the flow process has the form shown in Fig. 1a; it intensifies as the Grashof number is increased. We assume here that $T_2 > T_1$. If $T_2 < T_1$, then the direction of flow is reversed in the closed streamlines.

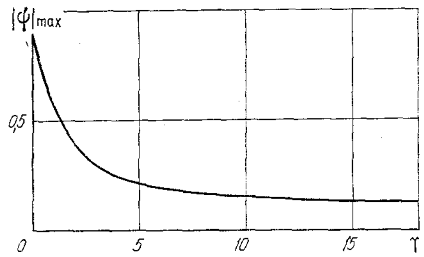

Fig. 2. Maximum value of the stream function versus the number γ.

With an increase in the number γ the flow diminishes, the center of circulation shifting toward the lower boundary. The velocity of the fluid at the surface decreases in this case (Fig. 1b). We note that the diminution of the flow intensity occurs very abruptly near the value $\gamma = 1$. Figure 2 shows the variation of $|\psi|_{max}$, which characterizes the flow intensity in the vortex, as the number γ is increased. It is seen that the flow intensity undergoes an abrupt increase as γ is decreased (for small values of this number).

In the calculations leading up to the results shown in Figs. 1 and 2 the period was taken to be one fourth the height of the layer (for $a/h = 0.4$). The flow pattern is not altered by increasing the period. Decreasing a/h for a fixed Grashof number diminishes the flow, but does not change it qualitatively.

The calculations have been carried out with a partition of the interval of integration into 10 and 20 segments along the y axis and into 20 and 40 segments along the x axis. The results agree within 3-5% error limits.

Thus, by a numerical solution of the complete Navier–Stokes equations in the Boussinesq approximation we have succeeded in determining the influence of stable stratification on the motion induced in a flat layer by a temperature inhomogeneity of the lower boundary. We have confirmed the occurrence of an abrupt decrease in the flow velocities as the stratification is enhanced. The generation of an intensely circulating flow in a stratified fluid layer as a result of an elevation of the temperature of part of the lower boundary requires that this temperature attain a value close to the temperature of the upper surface.

NOTATION

t, time; h, height of the layer; a, length of the section of the lower boundary with temperature T_2; x, y, Cartesian coordinates; v_x, v_y, velocity components; p, pressure; T, temperature; ν, kinematic viscosity; ρ, density; k, thermal conductivity; β, coefficient of cubical expansion; g, acceleration of gravity; T_1, T_2, T_3, temperatures of different parts of the boundaries; u, v, dimensionless velocity components; P, dimensionless pressure; ϑ, dimensionless temperature; $\gamma = (T_3-T_1)/(T_2-T_1)$.

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